Numerical Approach for Solving Rigid Spacecraft Minimum Time Attitude Maneuvers

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The minimum time attitude slewing motion of a rigid spacecraft with its controls provided by bounded torques and forces is considered. Instead of the slewing time, an integral of a quadratic function of the controls is used as the cost function. This enables us to deal with the singular and nonsingular problems in a unified way. The minimum time is determined by sequentially shortening the slewing time. The two-point boundary-value problem is derived by applying Pontryagin's maximum prinicple to the system and solved by using a quasilinearization algorithm. A set of methods based on the Euler's principal axis rotation is developed to estimate the unknown initial costates and the minimum slewing time as well as to generate the nominal solutions for starting this algorithm. It is shown that one of the four initial costates associated with the quaternions can be arbitrarily selected without affecting the optimal controls and thus simplifying the computation. Several numerical tests are presented to show the applications of these methods.

Introduction

THE problems of large-angle attitude maneuvers of a spacecraft have gained much considerations in recent years.[1-9] In these researches, the configurations of the spacecraft considered are 1) completely rigid, 2) a combination of rigid and flexible parts, or 3) gyrostat-type systems. The performance indices usually include minimum torque integration, power criterion, and frequency-shaped cost functionals, etc. Also, some of these investigations use feedback control techniques. In this paper, the minimum time attitude slewing control problem of a rigid spacecraft is considered.

In Ref. 2, the problem of the rapid torque-limited slewing of the rigidized Spacecraft Control Laboratory Experiment (SCOLE)[1] about a single axis (x axis) is considered. The control torque about this axis is of a bang-bang type or a bang-pause-bang type. The control laws are developed based on a simplified model of the SCOLE and then used on the practical model (with nonzero products of inertia); hence, this leads to large error of the attitude after the slewing. Also it seems that no details were given for the controls about the other two axes (y, z).

In the present paper, the optimal control theory (maximum principle) is applied to the slewing motion of a general rigid spacecraft (including the rigidized SCOLE, without simplification). The slewing motion need not be restricted to a single-axis slewing. The computational procedure based on a quasilinearization algorithm is developed to solve the resulting two-point boundary-value problem.

Euler Rotation and State Equations

The attitude of a rigid spacecraft can be described by either a quaternion $q = [q_0\ q_1\ q_2\ q_3]^T$, which satisfies a constraint equation, $q^T q = 1$, or a direction cosine matrix C:

$$C = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & (q_0^2 + q_2^2 - q_3^2 - q_1^2) & 2(q_2 q_3 + q_0 q_1) \\ 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & q_0^2 + q_3^2 - q_1^2 - q_2^2 \end{bmatrix} \tag{1}$$

It is known that a single-axis rotation of the spacecraft can also be expressed by a quaternion

$$q_0 = \cos(\theta/2); \qquad q_i = e_i \sin(\theta/2), \qquad i = 1,2,3 \tag{2}$$

where θ is the angle of rotation, and e_i are the direction cosines of the rotation axis.

The Euler rotation theorem tells us that an arbitrary orientation of a rigid body can be realized by rotating it about a principle axis (eigenaxis) through a certain angle from its initial position. The desired rotation quaternion q between the initial position $q(0)$ and the final orientation $q(t_f)$ can be obtained by the following equation:

$$\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_{00} & q_{10} & q_{20} & q_{30} \\ -q_{10} & q_{00} & q_{30} & -q_{20} \\ -q_{20} & -q_{20} & q_{00} & q_{10} \\ -q_{30} & q_{20} & -q_{10} & q_{00} \end{bmatrix} \begin{bmatrix} q_{0f} \\ q_{1f} \\ q_{2f} \\ q_{3f} \end{bmatrix} \tag{3}$$

where the second subscripts 0 and f represent the initial time and final time, respectively. The angle of rotation and the unit vector $\boldsymbol{e} = [e_1\ e_2,\ e_3]^T$, along this eigenaxis, are

$$\theta^* = 2\cos^{-1} q_0; \qquad e_i = q_i / \sqrt{1 - q_0^2}, \qquad i = 1,2,3 \tag{4}$$

The equations of motion of a rigid spacecraft are

$$\dot{q} = (1/2)\underline{\tilde{\omega}} q \tag{5}$$

$$I\dot{\omega} = \tilde{\omega} I \omega + Bu \tag{6}$$

where $\omega = [\omega_1\ \omega_2\ \omega_3]^T$ is the angular velocity vector, B is a $3 \times n$

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control influence matrix, $u = [u_1\, u_2\, u_3, \ldots, u_n]^T$ is the control torque and force vector, and

$$I = \begin{bmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{12} & I_{22} & -I_{23} \\ -I_{13} & -I_{23} & I_{33} \end{bmatrix}$$

$$\tilde{\omega} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}, \qquad \underline{\tilde{\omega}} = \left[\begin{array}{c|c} 0 & -\omega^T \\ \hline \omega & \tilde{\omega} \end{array}\right]$$

Premultiplied by the inverse of I, Eq. (6) can be rewritten as

$$\dot{\omega} = I^{-1}\tilde{\omega}I\omega + I^{-1}Bu \tag{7}$$

The boundary conditions for the states q and ω are given as

$$q(0), \quad \omega(0); \quad q(t_f), \quad \omega(t_f) \tag{8}$$

Optimal Control

Conventionally, the time optimal problem involved here is to seek a solution of Eqs. (5) and (7) satisfying the boundary conditions [Eq. (8)] and minimizing the slewing time:

$$t_f = \textstyle\int_0^{t_f}(1)\, \mathrm{d}t$$

The control variables must satisfy the constraints

$$|u| \le u_{i\max} = u_{ib}, \quad i = 1,2,\ldots,n \tag{9}$$

The Hamiltonian for this problem can be written as

$$H = 1 + (1/2)p^T\underline{\tilde{\omega}}q + r^T(I^{-1}\tilde{\omega}I\omega + I^{-1}Bu)$$

where p and r are the costate vectors associated with q and ω, respectively. They satisfy the necessary conditions

$$\dot{p} = -(\partial H/\partial q), \qquad \dot{r} = -(\partial H/\partial \omega) \tag{10}$$

By Pontryagin's maximum principle, the optimal control minimizing H can be determined by

$$u_i = -u_{ib}\mathrm{sgn}(B^TI^{-1}r)_i, \qquad i = 1,2,\ldots,n \tag{11}$$

which are of the bang-bang type, except for the singular case in which $(B^TI^{-1}r)_i = 0$ over some nonzero time intervals. For the nonsingular bang-bang optimal control problem, a shooting method[10] was tried and failed due to the nonexistence of the inverse of a partial differential matrix.

The singular problem does occur if a special case is considered in the following. When B is a 3×3 unit matrix and I is a diagonal inertia matrix, the control in Eq. (11) is simplified as

$$u_i = (u_{ib}/I_{ii})\mathrm{sgn}(r_i), \qquad i = 1,2,3 \tag{12}$$

If the state boundary conditions are such that a single principal axis (say, about axis 1) slewing is desired, then a solution satisfying the necessary conditions [Eq. (10)] and Eqs. (5) and (7) for this problem can be obtained as

$$q_0(0) \neq 0, \quad q_1(0) \neq 0, \quad q_2(0) = q_3(0) = 0, \quad \omega_1(0) \neq 0$$
$$\omega_2(0) = \omega_3(0) = 0, \quad q_0 \neq 0, \quad q_1 \neq 1, \quad q_2 = q_3 = 0$$
$$p_0 \neq 0, \quad p_1 \neq 0, \quad p_2 = p_3 = 0, \quad \omega_1 \neq 0$$
$$\omega_2 = \omega_3 = 0, \quad r_1 \neq 0, \quad r_2 = r_3 = 0$$
$$u_1 = -(u_{1b}/I_{11})\mathrm{sgn}(r_1), \quad u_2 = u_3 = 0$$

The solution $r_2 = r_3 = 0$ implies a singular control problem because u_2 and u_3 cannot be determined by Eq. (12). The possible existence of a singular solution in the general minimum time problem suggests that a unified method is needed to handle both singular and nonsingular cases.

In some papers,[3,8] the integral of the sum of the squares of the torque components has been successfully used as a cost function, where there is no constraint on the control and the minimum time is not required. However, if some constraints on the control are added to this problem and the total slewing time is shortened sequentially, this problem may approach the minimum time control problem. These considerations motivate a successive approximation approach to solve the minimum time control problem. In this approach, an integral of a quadratic function of the controls is formally used as a cost function, i.e.,

$$J = \frac{1}{2}\int_0^{t_f} u^TRu\, \mathrm{d}t \tag{13}$$

where R is a proper weighting matrix. It has been shown[3,8] that for the case of rest-to-rest slewing with only three control inputs involved and with a longer slewing time used, the controls are approximately linear functions of time and do not reach their saturation levels. Therefore, when t_f is shortened, some of the controls can be expected to reach their bounds and contribute more effort to the slewing. By the successive shortening of t_f, a particular value, $t_f{}^*$, called the minimum time, can be obtained, during which either some (singular case) or all (nonsingular case) of the controls are of the bang-bang type.

Apparently, in the aforementioned approach, it is unnecessary to determine in advance whether the problem is singular, and there is no need to determine the switching points as required in some other methods.[10]

Necessary Conditions

The Hamiltonian for the system [Eqs. (5), (7), and (13)] is then

$$H = (1/2)u^TRu + p^T\underline{\tilde{\omega}}q + r^T(I^{-1}\tilde{\omega}I\omega + I^{-1}Bu) \tag{14}$$

where the costates, as before, satisfy the following necessary conditions for minimizing J:

$$\dot{p} = -(\partial H/\partial q) \quad \text{or} \quad \dot{p} = (1/2)\underline{\tilde{\omega}}p \tag{15}$$

$$\dot{r} = -(\partial H/\partial \omega) \quad \text{or} \quad \dot{r} = g(\omega,r) + (1/2)[q]p \tag{16}$$

where $q(\omega,r)$ is a 3×1 vector function of ω and r, and its detailed form can be found in Appendix A; $[q]$ is a 3×4 matrix

$$[q] = \begin{bmatrix} q_1 & -q_0 & -q_3 & q_2 \\ q_2 & q_3 & -q_0 & -q_1 \\ q_3 & -q_2 & q_1 & -q_0 \end{bmatrix}$$

The initial values of p and r are unknowns.

The weighting matrix R in Eq. (13) is chosen as

$$R = B^TB \tag{17}$$

which is generally an $n \times n$ semi-positive-definite matrix, because the rank of the $3 \times n$ matrix, B, is assumed to be 3. Weighting matrices other than that given by Eq. (17) may also be possible candidates.

From the necessary conditions $(\partial H/\partial u) = 0$, we have

$$Ru + B^TI^{-1}r = 0$$

or

$$B^TBu = -B^TI^{-1}r \tag{18}$$

Premultiplying both sides of Eq. (18) by $(BB^T)^{-1}B$, one obtains

$$Bu = -I^{-1}r$$

By using the pseudoinverse of the matrix B, B^+, one can get u:

$$u = -B^+I^{-1}r = -B^T(BB^T)^{-1}(I^{-1}r) \tag{19}$$

The control laws are then[11]

$$u = -u_{ib}\text{sgn}(B^+I^{-1}r)_i \qquad \text{if} \qquad |(B^+I^{-1}r)_i| \geq u_{ib} \tag{20a}$$

or

$$u_i = -(B^+I^{-1}r)_i \qquad \text{if} \qquad |(B^+I^{-1}r)_i| < u_{ib}, \quad i = 1,2,\ldots,n \tag{20b}$$

note that, when B is a 3×3 nonsingular matrix, $B^+ = B^{-1}$.

Linear Relation Between q and p

Before starting to solve the two-point boundary-value problem, it is useful to consider a relationship between q and p. It is already pointed out in Ref. 9 that

$$p \neq \rho q$$

where ρ is an arbitrary constant. However, one can find out that there does exist a linear relation between p and q in this problem:

$$p(t) = Dq(t) \tag{21}$$

where D is a 4×4 constant matrix. To determine the constant elements of D, Eq. (21) is substituted into the differential equation for p in Eq. (15), with the result

$$D\dot{q} = (1/2)\underline{\tilde{\omega}}Dq \tag{22}$$

This $\dot{q}$ in Eq. (22) is then replaced by Eq. (5);

$$D(1/2)\underline{\tilde{\omega}}q = (1/2)\underline{\tilde{\omega}}Dq \tag{23}$$

Since Eq. (23) is valid for arbitrary q, one has

$$D\underline{\tilde{\omega}} = \underline{\tilde{\omega}}D$$

This relation is true for arbitrary values of ω only when D has the following form:

$$D = \begin{bmatrix} d_0 & -d_1 & -d_2 & -d_3 \\ d_1 & d_0 & -d_3 & d_2 \\ d_2 & d_3 & d_0 & -d_1 \\ d_3 & -d_2 & d_1 & d_0 \end{bmatrix} \tag{24}$$

and tnese constants d_0, d_1, d_2, and d_3 can be determined by setting $t = 0$ in Eq. (21). With the use of Eq. (24), the relation (21) can be rewritten as

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{25}$$

Substituting Eq. (25) into Eq. (16) results in

$$\dot{r} = g(\omega, r) - (1/2)Cd \tag{26}$$

where $d = [d_1\ d_2\ d_3]^T$, C is just the attitude matrix [Eq. (1)]. It can be seen from Eq. (26) that r is independent of d_0. It is also true that u is independent of d_0 because u depends only on r in Eqs. (20). This means that the arbitrary selection of d_0 yields the same extremum control, u. It is noted that a special choice of d_0 can lead to the equivalent conditions considered in Refs. 3 and 9. With the use of Eq. (25), one can get

$$d_0^2 + d_1^2 + d_2^2 + d_3^2 = p^T(0)p(0) \tag{27}$$

Since the choice of d_0 is independent of the choice of d_1, d_2, and d_3, a minimum value of the left side (hence, the right side) of Eq . (27) is reached when $d_0 = 0$. This is the solution considered in Ref. 3. Also, from Eq. (25) one can get

$$d_0 = p^T(0)q(0)$$

When $d_0 = 0$, this equation gives a constraint on $p(0)$. In Ref. 9 it is suggested that this constraint be used in the numerical iterations. But the choice of d_0 other than zero is a more general result for this problem. It is not necessary to keep $d_0 = 0$ in each step of the computation. It is enough to keep one element of $p(0)$ unchanged, which is easier to use than the approach suggested in Ref. 9, especially when $q(0) \neq [1\ 0\ 0\ 0]^T$.

Initial Values of Costates and the Slewing Time

Since the Euler rotation brings the attitude of the spacecraft from an initial quaternion to a final required quaternion through a simple rotation, it may take less time and consume less energy; it is reasonable to choose this rotation as a candidate for the starting solution of the iteration and hope that the optimal slewing is near the Euler rotation. This rotation will be called the "expected rotation," which is determined only by $q(0)$ and $q(t_f)$.

The angular velocity and its derivatives for the Euler rotation can be expressed as

$$\bar{\omega} = \dot{\theta}e, \qquad \dot{\bar{\omega}} = \ddot{\theta}e, \qquad \ddot{\bar{\omega}} = \dddot{\theta}e \tag{28}$$

where $\theta(t)$ is the rotation angle and $e = [e_1\ e_2\ e_3]^T$ is a unit vector along the rotation axis (eigenaxis) which is determined by Eq. (4). Considering the analytical solution about a single principal axis maneuver in Ref. 3, θ can be defined the same way about e:

$$\theta(t) = \theta(0) + \dot{\theta}(0)t + (1/2)\ddot{\theta}(0)t^2 + (1/6)\dddot{\theta}(0)t^3 \tag{29}$$

where $\theta(0)$, $\dot{\theta}(0)$, $\ddot{\theta}(0)$, and $\dddot{\theta}(0)$ can be determined from the boundary conditions of $\theta(t)$ and $\dot{\theta}(t)$ at $t = 0$, and $t = t_f$.

Without loss of generality, one can choose

$$\theta(0) = 0, \qquad \theta(t_f) = \theta^* \tag{30a}$$

where θ^* is given in Eq. (4), and

$$\dot{\theta}(0) = \dot{\theta}_0, \qquad \dot{\theta}(t_f) = \dot{\theta}_f \tag{30b}$$

The value of $\dot{\theta}_0$ in Eq. (30b) needs to be determined from a given initial angular velocity vector $\omega(0)$. Generally, this vector will not coincide with $\bar{\omega}(0)$, the angular velocity of the Euler rotation at $t = 0$, defined in Eq. (28); therefore, a difference vector ϵ exists between them:

$$\epsilon = \dot{\theta}_0 e - \omega(0)$$

Since only an approximate starting solution of the quasilinearization method is needed, it is enough to choose a $\dot{\theta}_0$ that minimizes $\epsilon^T\epsilon$. By differentiating $\epsilon^T\epsilon$ with respect to $\dot{\theta}_0$ and noting that $e^Te = 1$, one can get

$$\dot{\theta}_0 = e^T\omega(0) \tag{31}$$

A similar derivation for $\dot{\theta}_f$ can be obtained.

For the special case of $\dot{\theta}_f = 0$, the substitution of Eqs. (30) into Eq. (29) will result in

$$\ddot{\theta}(0) = (6\theta^*/t_f^2) - (4\dot{\theta}_0/t_f) \tag{32a}$$

$$\dddot{\theta}(0) = -(12\theta^*/t_f^3) + (6\dot{\theta}_0/t_f^2) \tag{32b}$$

To approximately determine the initial values of p and r, Eqs. (7) and (26) are needed. By substituting u in Eq. (19) into Eq. (7) and solving for r, one can get

$$r = I\tilde{\omega}I\omega - I^2\dot{\omega} \tag{33}$$

$$\dot{r} = (d/dt)(I\omega I\tilde{\omega}) - I^2\ddot{\omega} \tag{34}$$

At the same time, Eq. (26) can be rewritten as

$$d = 2C^T[g(\omega,r) - \dot{r}] \tag{35}$$

Replacing ω in Eqs. (33–35) by the relations (28–32) at $t = 0$, one can get the approximate values of $r(0)$ and d where d is defined after Eq. (26). The $p(0)$ can be determined by letting one of its elements equal a constant (say, $p_0(0) = \text{const}$, if $q(0) = [1\ 0\ 0\ 0]^T$) and by using Eq. (25) to solve for d_0 and other elements of $p(0)$.

The starting solution needed in the quasilinearization algorithm may be obtained by integrating the differential Eqs. (5), (7), (15), (16), and (20) using the preceding initial conditions $p(0)$ and $r(0)$, as well a $q(0)$ and $\omega(0)$.

Initial Value of t_f

Generally, to obtain the minimum time, one can always choose a longer slewing time t_f at the beginning of the algorithm and shorten it sequentially thereafter. But this may take more time, especially when it is not known how far the initial choice is from the real minimum time. Therefore, a good initial value of t_f being close to and larger than the minimum value is desired. For simplicity, only an estimation procedure of t_f for the case in which B is a 3 × 3 unit matrix is discussed here. Suppose the slewing motion is an Euler rotation about a vector e through an angle $\theta(t)$. By using the relations for ω in Eq. (28) into Eq. (6), one can get

$$Ie\ddot{\theta} = \dot{\theta}^2\tilde{e}Ie + u$$

which can be expressed as the following three similar equations

$$a_i\ddot{\theta} = b_i\dot{\theta}^2 + c_i\tau_i, \qquad i = 1,2,3 \tag{36}$$

where a_i and b_i are constants, $c_i = u_{ib}$, and τ_i is the normalized control about the ith body axis and

$$|\tau_i| \le 1, \qquad i = 1,2,3$$

The three equations of Eq. (36) must be simultaneously valid for the same $\theta(t)$. Each of them with the boundary conditions [Eqs. (30)] can be considered as a minimum time control problem and solved analytically to obtained a minimum time (Appendix B). Since each of the minimum times for the associated equation means a lower bound of time during which the equation is solvable (regardless of whether this equation is treated as a minimum time control problem), the largest one of these minimum times should be chosen as the initial value of t_f used in the computation.

A computation procedure has been developed that contains a series of cycles. The slewing time is chosen at the beginning of each cycle and fixed throughout the cycle. During each cycle, a quasilinearization algorithm called the method of particular solutions[12] is used to solve the linearized state and costate equations. If this algorithm converges, a check is then made as to whether some (singular case) or all (nonsingular case) of the controls are of the bang-bang type. If so, this slewing time is designated the minimum time. If not, the assumed t_f should be shortened and the next cycle begins.

The numerical experience of using this procedure tells us that, for each cycle, the slewing time cannot be made less than a certain value; in particular, it cannot be made less than the real minimum time. Otherwise, the algorithm in each cycle will not converge. The closer the t_f is to the real minimum time, the less shortening is required for the t_f assumed in the previous cycle.

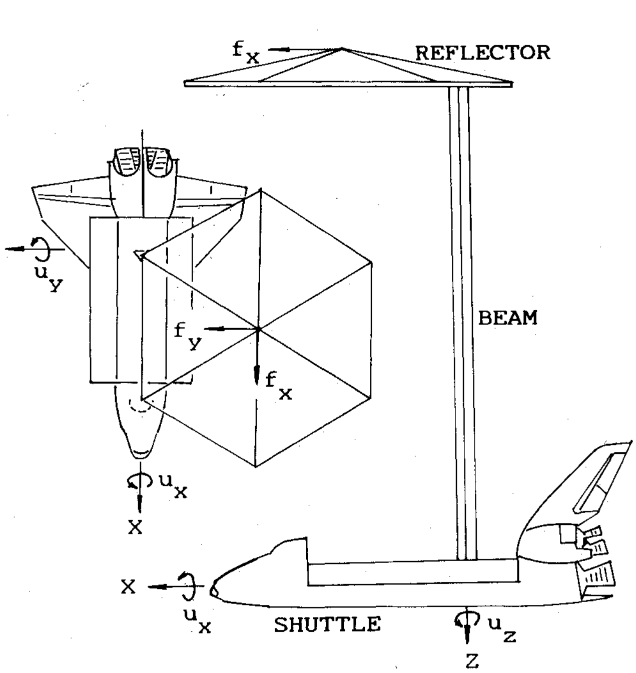

Fig. 1 Spacecraft control laboratory experiment configuration (SCOLE).

Numerical Results

The methods described in the previous sections are applied to the SCOLE slewing motion. Figure 1 shows the SCOLE configuration. It is composed of a Space Shuttle and a large reflecting antenna. The antenna is attached to the Shuttle by a flexible beam. Since only the motion of the rigid SCOLE is considered in this paper, the flexibility of the beam is ignored. The X, Y, and Z axes are the Shuttle axes corresponding to he roll, pitch, and yaw axes, respectively. The controls considered in this paper include three moments ($u_x = u_1$, $u_y = u_2$, $u_z = u_3$) about the X, Y, and Z axes and two forces ($f_x = u_4$, $f_y = u_5$) applied at the center of the reflector in the X and Y directions. The inertia parameters of the SCOLE and the saturation levels of the controls are[1]

$$I_{11} = 1132508, \quad I_{22} = 7007447, \quad I_{33} = 7113962$$

$$I_{12} = -7555, \quad I_{23} = 115202, \quad I_{13} = 52293 \text{ slug-ft}^2$$

$$u_{ib} = 10000 \text{ ft-lb}, \quad i = 1,2,3, \quad u_{ib} = 800 \text{ lb}, \quad i = 4,5$$

The associated control influence matrix B in Eq. (6) is

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 130 \\ 0 & 1 & 0 & -130 & 0 \\ 0 & 0 & 1 & 32.5 & 18.75 \end{bmatrix}$$

Some numerical results are presented in the following cases.

1) The singular case discussed in the previous sections for the SCOLE configuration without offset and for a symmetrical Shuttle $I_{12} = I_{23} = 0$. For this case only the three Shuttle con-

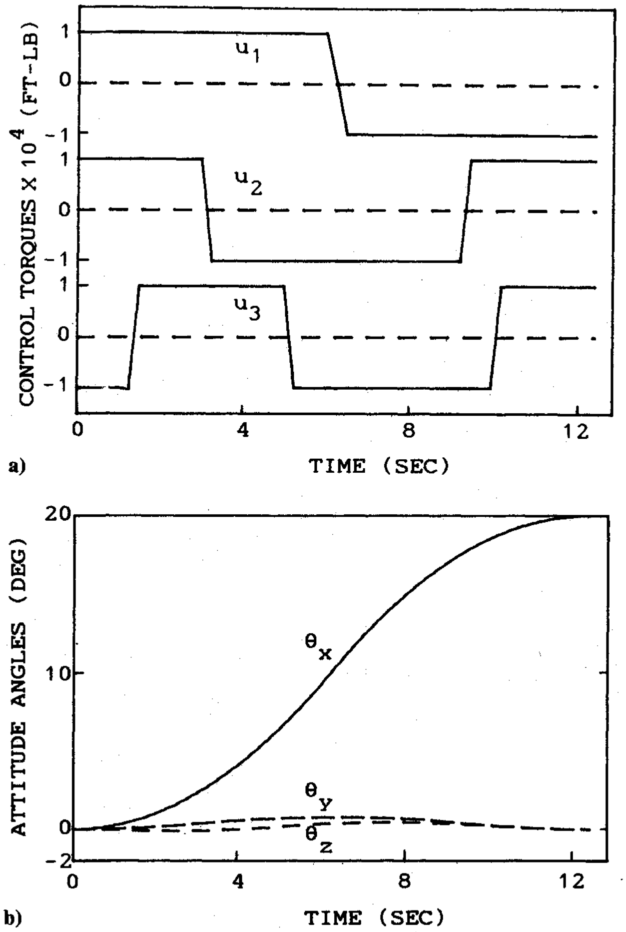

Fig. 2 The X-axis slewing; $t_f = 12.563034$ s.

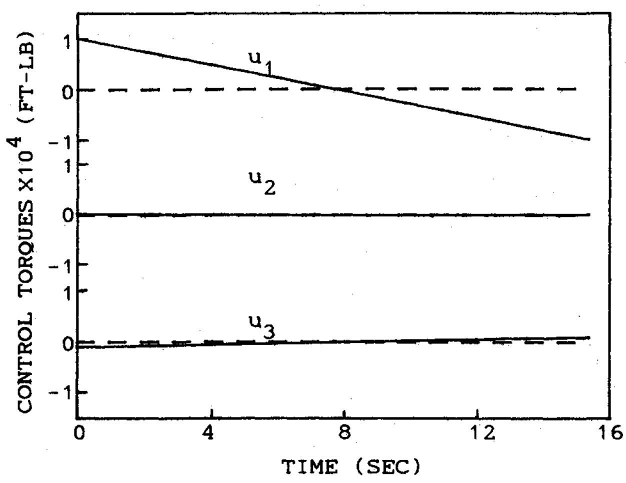

Fig. 3 The X-axis slewing; $t_f = 15.37$ s.

trol torquers are used. The boundary conditions of the attitude are such that the "expected rotation" is a single principal axis rotation. The computations show that, when the slewing time approaches the minimum time, the control about the the slewing axis approaches the bang-bang type. The other two controls remain zero, and there is no indication that they are going to make contributions to speed up the slewing. This result may imply that the singular solution is the time optimal solution for this case; otherwise, additional control effort should participate in the slewing and a smaller slewing time should be obtained by using this algorithm.

2) The example of Ref. 3 is computed to test the method of determination of the initial costates. The results show that the guessed initial costates are very close to their converged values. To obtain the converged values (to seven digits), only five iterations are needed in this computation.

3) The nondiagonal inertia matrix of the SCOLE and only three controls (u_1, u_2, u_3) are used in this case. The expected rotation is a 20-deg, rest-to-rest rotation about one of the three spacecraft axes. Only the results for the "X-axis slewing" are given here because the results for the Y-axis and Z-axis slewings are similar. Figure 2 shows the time histories of the controls and attitude angles (1-2-3 Euler angles) for this maneuver. Because of the nonzero offset of the inertia distribution of the SCOLE configuration, the controls u_y and u_z are no longer zero as they were in the singular case (case 1), but are now of the bang-bang type. The initial estimation of the minimum slewing time t_f, obtained using the method discussed in the previous section, is $t_f^{(0)} = 12.5749$ s, which is very close to the minimum time $t_f^* = 12.563034$ s obtained in our computation.

Figure 3 shows the control torques for the same X-axis slewing but with a slewing time $t_f = 15.37$ s, which is 2.8 s more

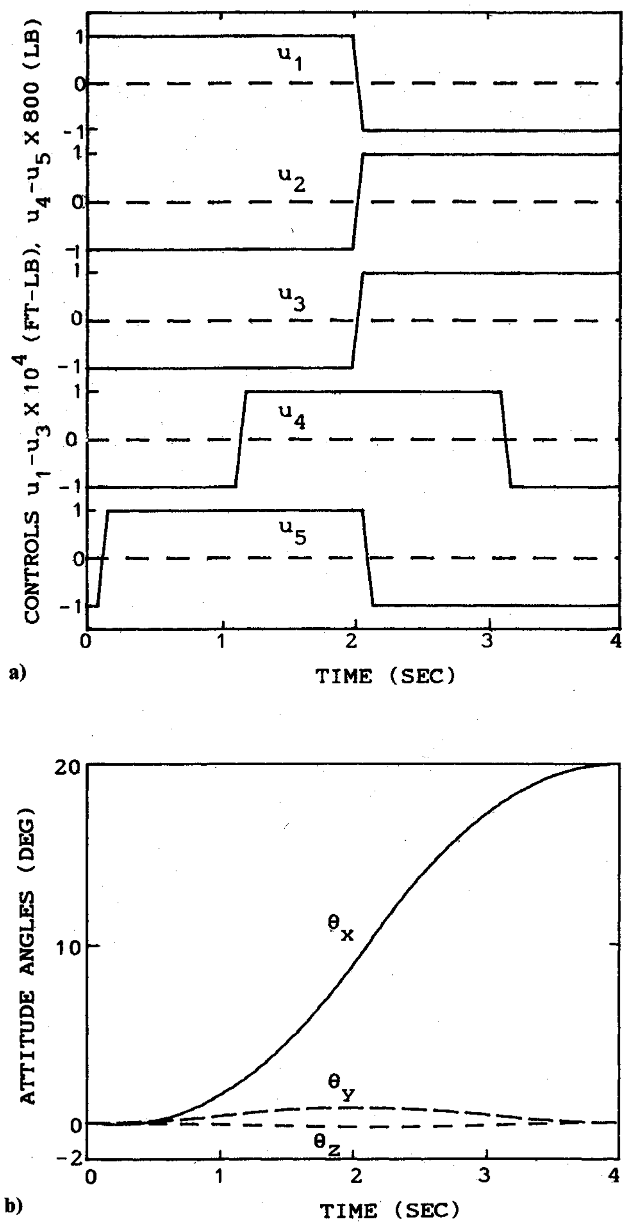

Fig. 4 The X-axis slewing; $t_f = 3.9805382$ s.

than the minimum time $t_f{}^*$. The controls are almost linear functions of time. The u_1 is less than the saturation level; u_2 and u_3 are near zero. By comparing Fig. 3 with Fig. 2a, one can see that much more control effort (approximate 50%) is saved by using a little longer slewing time. Another feature of using a longer slewing time in the computation is that it needs fewer iterations for convergence than by using a shorter slewing time. These properties imply that, in the practical application of this problem, it is not necessary to seek exactly the $t_f{}^*$ and the associated extremum controls. It may be enough to know the approximate values of $t_f{}^*$ and the controls.

4) Following case 3, two additional controls, u_4 and u_5, corresponding to the thrusters on the reflector was used. Figure 4 shows the control and attitude angles for the X-axis slewing. The minimum time $t_f{}^* = 3.9805382$ s is greatly shortened compared with case 3. Figure 5 shows the controls and attitude angles for the Z-axis slewing. The minimum time is $t_f{}^* = 15.1441$ s. Unlike the case for the X-axis slewing, the attitude angle θ_x experiences a larger amplitude, though the expected rotation is about the Z axis. This phenomenon is due to the unsymmetric distributions of inertia about the X and Y axes. The closer the slewing is to the minimum time, the larger the amplitude of the θ_x.

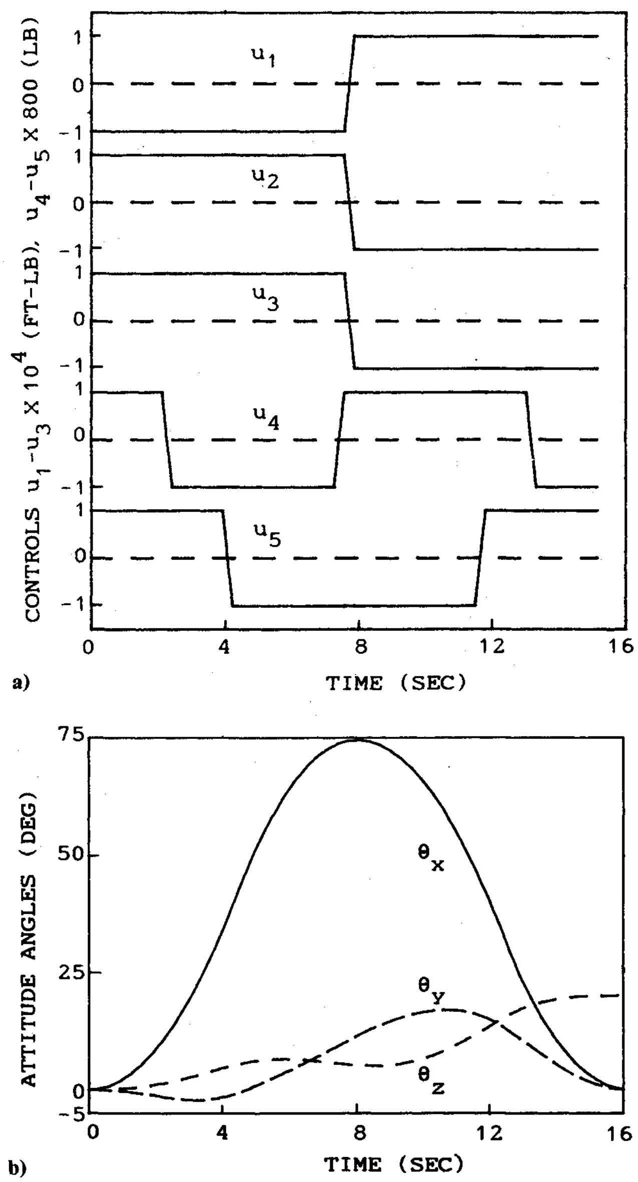

Fig. 5 The Z-axis slewing; $t_f = 15.1441$ s.

5) A general case is considered. Suppose the SCOLE is in an Earth orbit and the line of sight is to be directed toward the center of the Earth. The orbit coordinate system (x,y,z) is shown in Fig. 6. The initial attitude of the spacecraft is assumed as follows: the Y axis coincides with the orbital y axis, and the angular difference between the X and x (or Z and z) axes is $\alpha = 7.897224212$ deg. The initial quaternion is then $q(0) = [\cos(\alpha/2)\ 0\ \sin(\alpha/2)\ 0]^T$. According to Ref. 1, the unit vector along the line of sight in the rigid SCOLE coordinate system is

$$\hat{R}_{LOS} = [0.1112447 \quad -0.2410302 \quad 0.9641209]^T$$

The direction cosines of the orbital z axis in the SCOLE system at the initial time are $\hat{z}_{/B} = [\sin\alpha\ 0\ \cos\alpha]^T$. The angle between $\hat{R}$ and $\hat{z}_{/B}$ at the initial time is $\theta_{LOS}(0) = \hat{R}_{LOS} \cdot \hat{z}_{/B} = 20$ deg. The eigenaxis of the expected rotation in the SCOLE system is determined by

$$e = (\hat{R}_{LOS} \times \hat{z}/B / |\hat{R}_{LOS} \times \hat{z}/B|$$

The final required attitude quaternion can be obtained by using Eqs. (3) and (4).

The estimated minimum time for the case where only three controls are involved in this maneuver is $t_f = 26.3487$ s. This value is very close to the converged value $t_f{}^* = 25.003175$ s. It would be interesting if this estimation is compared with the result by solving the following classical minimum time control problem:

$$I\ddot{\theta} = u, \qquad |u| \leq u_{\max}$$

where I is the moment of inertia about the principal line, u is the torque about that line, and θ is the angle of rotation. For the present case, the result obtained from the second method is $t_f = 19.58$ s. A possible explanation for the larger difference here is that the classical problem greatly simplifies the inherent three-dimensional nonlinear dynamics associated with the general SCOLE configuration.

Figure 7 shows the controls and attitude angles for assumption 5 where u_4 and u_5 are also used. The minimum slewing time is obtained as $t_f{}^* = 8.691397$ s. The θ_{LOS} in Fig. 7b is the angle between the line of sight and the line of the target direction (from the spacecraft to the center of the Earth).

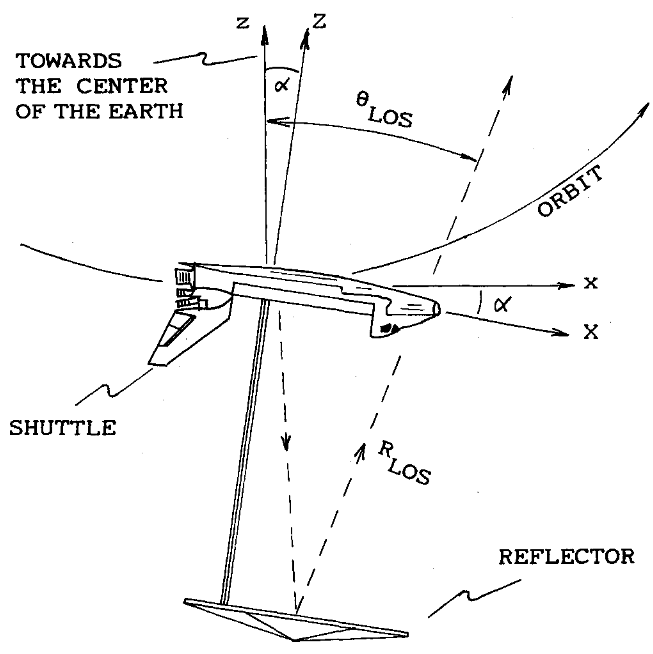

Fig. 6 Attitude of the SCOLE showing antenna line of sight.

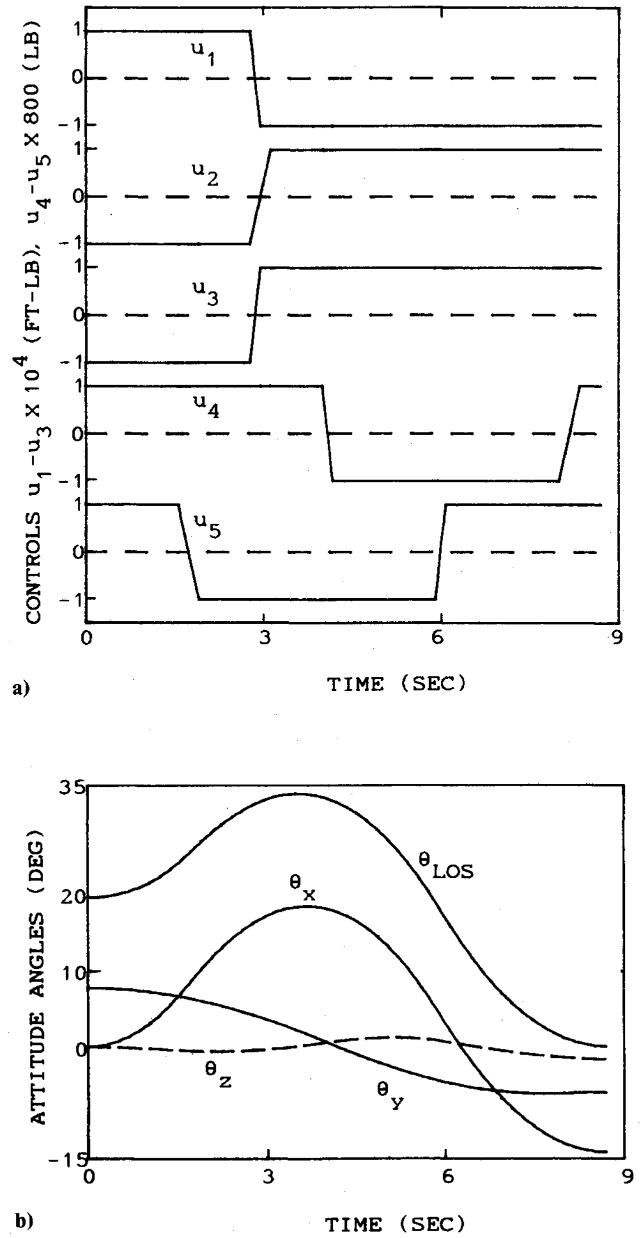

Fig. 7 Example of line of sight slewing for SCOLE; $t_f = 8.691397$ s.

Conclusions

A useful numerical solution procedure for the minimum time attitude maneuver control problem of a rigid spacecraft has been developed and successfully applied to some practical examples. It can handle both the singular and nonsingular cases. The examples show that the estimation methods used here for the initial costates and the minimum slewing time are quite useful. The control profiles obtained in this paper may be useful for further research.

Appendix A: The Term $g(\omega, r)$ in Eq. (16)

The term $I^{-1}\tilde{\omega}I\omega$ in Eq. (7) can be expressed as

$$I^{-1}\tilde{\omega}I\omega = [F{:}G]\bar{\bar{\omega}}$$

and

$$\bar{\bar{\omega}} = [\omega_1^2 \; \omega_2^2 \; \omega_3^2 \; \omega_2\omega_3 \; \omega_3\omega_1 \; \omega_1\omega_2]^T$$

where F and G are 3×3 matrices whose elements are constants associated with the inertia parameters of the spacecraft. Then the term $r^T I^{-1}\tilde{\omega}I\omega$ of the Hamiltonian H in Eq. (14) has the form

$$h = r^T I^{-1}\tilde{\omega}I\omega = [f_1 \, f_2 \, f_3 \, f_4 \, f_5 \, f_6]^T \bar{\bar{\omega}}$$

where f_i are

$$[f_1 \, f_2 \, f_3]^T = F^T r, \qquad [f_4 \, f_5 \, f_6]^T = G^T r$$

The term $g(\omega, r)$ in Eq. (16) can be obtained by

$$g(\omega,r) = -(\partial h/\partial\omega) = -\begin{bmatrix} 2f_1 & f_6 & f_5 \\ f_6 & 2f_2 & f_4 \\ f_5 & f_4 & 2f_3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

Appendix B: Solution of Eq. (36)

Equation (36) can be rewritten as

$$a_i\ddot{\theta} = b_i\dot{\theta}^2 + c_i\tau_i$$

For simplicity, only the solutions for the following boundary conditions are considered here:

$$\theta(0) = 0, \qquad \dot{\theta}(0) = 0; \qquad \theta(t_f) = \theta^*, \qquad \dot{\theta}(t_f) \equiv 0 \tag{B1}$$

Suppose $a_i \neq 0$, $b_i \neq 0$ and let $b = b_i/a_i$, $c = c_i/a_i$ (suppose $c > 0$), Eq. (36) can be rewritten as

$$\ddot{\theta} = b\dot{\theta}^2 + c\tau \tag{B2}$$

Since the control for this problem is of a bang-bang type with only once switching point, by integrating Eq. (B2) and using Eq. (B1), one can get

$$\dot{\theta} = [c(e^{2b\theta} - 1)/b]^{1/2} \qquad \text{for} \qquad \tau = 1 \tag{B3}$$

$$\dot{\theta} = [c(1 - e^{2b(\theta - \theta^*)})/b]^{1/2} \qquad \text{for} \quad \tau = -1 \tag{B4}$$

By equating Eqs. (B3) and (B4), one can get $\theta = \theta_s$ and $\dot{\theta} = \dot{\theta}_s$ at the switching point $t = t_s$:

$$\theta_s = (1/2b)\,\ell n[2/(1 + e^{-2b\theta^*})]$$

$$\dot{\theta}_s = [c(e^{2b\theta_s} - 1)/b]^{1/2}$$

Finally, by integrating Eqs. (B3) and (B4) and using Eq. (B1), one can get

$$t_s = \cosh^{-1}(e^{-b\theta_s})/\sqrt{-bc}, \qquad b < 0$$

or

$$t_s = [(\pi/2) - \sin^{-1}(e^{-b\theta_s})]/\sqrt{-bc}, \qquad b > 0$$

and

$$t_f = t_s + [(\pi/2) - \sin^{-1}(e^{b(\theta^* - \theta_s)})]/\sqrt{-bc}, \qquad b < 0$$

or

$$t_f = t_s + \cosh^{-1}(e^{b(\theta^* - \theta_s)})/\sqrt{bc}, \qquad b > 0$$

For the case $\dot{\theta}(0) \neq 0$, similar solutions can be obtained.

Acknowledgment

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